The Rasmussen invariant of a homogeneous knot

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Abstract

A homogeneous knot is a generalization of alternating knots and positive knots. We determine the Rasmussen invariant of a homogeneous knot. This is a new class of knots such that the Rasmussen invariant is explicitly described in terms of its diagrams. As a corollary, we obtain some characterizations of a positive knot. In particular, we recover Baader's theorem which states that a knot is positive if and only if it is homogeneous and strongly quasipositive.

1 Introduction

In [25], Rasmussen introduced a smooth concordance invariant of a knot K by using the Khovanov-Lee theory (see [15] and [16]), now called the Rasmussen invariant $s(K)$. This gives a lower bound for the four ball genus $g_*(K)$ of a knot K as follows.

$$|s(K)| \leq 2g_*(K). \tag{1.1}$$

This lower bound is very powerful and it enables us to give a combinatorial proof of the Milnor conjecture on the unknotting number of a torus knot. Our motivation for studying the Rasmussen invariant is to describe $s(K)$ in terms of a given diagram of a knot K to better understand $g_*(K)$. From this point of view, some estimations of the Rasmussen invariant are known (Plamenevskaya [24], Shumakovitch [30] and Kawamura [12]. See also Stoimenow [32]).

Let $O_+(D)$[1] and $O_-(D)$ be the numbers of connected components of the diagrams which is obtained from D by smoothing all negative and positive crossings of D, respectively. Recently, Kawamura [13] and Lobb [20] independently obtained a more sharper estimation for the Rasmussen invariant as follows.

Theorem 1.1 ([13] and [20]). *Let D be a diagram of a knot K. Then*

$$w(D) - O(D) + 2O_+(D) - 1 \leq s(K),$$

where $\omega(D)$ denotes the writhe of D (i.e. the number of positive crossings of D minus the number of negative crossings of D) and $O(D)$ denotes the number of the Seifert circles of D.

Let $\Delta(D) = O(D) + 1 - O_+(D) - O_-(D)$ (a graph theoretical interpretation of $\Delta(D)$ due to Lobb is given in Section 3). In addition to Theorem 1.1, Lobb [20] showed that if $\Delta(D) = 0$, then $s(K) = w(D) - O(D) + 2O_+(D) - 1$.

Our motivation for this paper is to study which diagrams D satisfy the condition $\Delta(D) = 0$. Lobb [20] showed that if D is positive, negative, alternating, or a certain braid diagram, then

[1] In [13] and [20], it was denoted by $l_0(D)$ and #components$(T^+(D))$ respectively.

$\Delta(D) = 0$. Note that these diagrams are all homogeneous (the definition is given in Section 2). In this paper, we show that if D is a homogeneous diagram of a knot, then $\Delta(D) = 0$ (the converse is also true. See Theorem 3.4) and our main result is to determine the Rasmussen invariant of a homogeneous knot. This is a new class of knots such that the Rasmussen invariant is explicitly described in terms of its diagrams.

Theorem 1.2. *Let D be a homogeneous diagram of a knot K. Then*

$$s(K) = w(D) - O(D) + 2O_+(D) - 1.$$

Ozsváth and Szabó [22] and Rasmussen [26] independently introduced another smooth concordance invariant of a knot K by using the Heegaard Floer homology theory, now widely known as the tau invariant $\tau(K)$. The Rasmussen invariant and tau invariant share some formal properties and these are closely related to positivity of knots. There are many notions of positivity (e.g. braid positive, positive, strongly quasipositive and quasipositive). We recall these notions of positivity in Section 3. Let D be a diagram of a knot K. Then Kawamura [13] also proved

$$w(D) - O(D) + 2O_+(D) - 1 \leq 2\tau(K).$$

Note that, if $\Delta(D) = 0$, $2\tau(K) = w(D) - O(D) + 2O_+(D) - 1$.[2] Therefore the corresponding result to Theorem 1.2 holds for the tau invariant. In particular, we obtain $\tau(K) = s(K)/2$ for a homogeneous knot K.

On the other hand, the Rasmussen invariant and tau invariant sometimes behave differently. It has been conjectured that $\tau = s/2$, however, Hedden and Ording [11] proved that the Rasmussen invariant and tau invariant are distinct (see also [19]). It may be worth remarking that the Rasmussen invariant is sometimes stronger than the tau invariant as an obstruction to a knot being smoothly slice ([11] and [19], see also [7]). This is the reason why we are more interested in the Rasmussen invariant rather than the tau invariant.

One can easily see that a braid positive knot is strongly quasipositive, however, it is not obvious whether a positive knot is strongly quasipositive. Nakamura [21] and Rudolph [28] independently proved that a positive knot is strongly quasipositive. Not all strongly quasipositive knots are positive. For instance, such examples are given by divide knots [27]. Rudolph [28] asked whether positive knots could be characterized as strongly positive knots with some extra geometric conditions. Several years later, Baader found that the extra condition is homogeneity. To be precise, Baader [2] proved that a knot is positive if and only if it is homogeneous and strongly quasipositive. As a corollary of Theorem 1.2, we obtain some characterizations of a positive knot.

Theorem 1.3. *Let K be a knot. Then* (1)–(4) *are equivalent.*
(1) *K is positive.*
(2) *K is homogeneous and strongly quasipositive.*
(3) *K is homogeneous, quasipositive and $g_*(K) = g(K)$.*
(4) *K is homogeneous and $\tau(K) = s(K)/2 = g_*(K) = g(K)$.*

In particular, we recover Baader's theorem. Note that our proof is 4-dimensional in the sense that we use concordance invariants, whereas Baader [2] used the Homflypt polynomial. As an immediate corollary of Theorem 1.3, we obtain the following.

Corollary 1.4. *Let K be a homogeneous knot. Then the following are equivalent.*
(1) *K is positive.*

[2] by using the fact that $-\tau(K) = \tau(\overline{K})$ for any knot K [22], where $\overline{K}$ denotes the mirror image of K.

(2) *K is strongly quasipositive.*
(3) *K is quasipositive and $g_*(K) = g(K)$.*
(4) $\tau(K) = s(K)/2 = g_*(K) = g(K)$.

It may be interesting to compare Corollary 1.4 and the following proposition by Hedden.

Proposition 1.5 ([10]). *Let K be a fibered knot. Then the following are equivalent.*
(1) *K is strongly quasipositive.*
(2) *K is quasipositive and $g_*(K) = g(K)$.*
(3) $\tau(K) = g_*(K) = g(K)$.

At a first glance, we wonder why similar results hold for fibered knots and homogeneous knots. However, it is not surprising since homogeneous knots are related to fiberedness. For instance, a knot which admits a homogeneous braid diagram is fibered (see Section 2 or Proposition 1.4 in [20]).

This paper is constructed as follows. In Section 2, we observe a geometric aspect of a homogeneous knot. In Section 3, we give a new characterization of a homogeneous diagram of a knot and determine the Rasmussen invariant of a homogeneous knot (Theorem 1.2). In Section 4, we recall some notions of positivity for knots and give some characterizations of a positive knot (Theorem 1.3). In Section 5, we propose a new approach to estimate the Rasmussen invariant of a knot.

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2 Geometric aspect of a homogeneous knot

Cromwell [4] introduced the notion of homogeneity for knots to generalize results on alternating knots. The notion of homogeneity is also defined for signed graphs and diagrams. For graph theoretical terminologies in this paper, we refer the reader the book of Cromwell [5].[3]

A graph is *signed* if each edge of the graph is labeled $+$ or $-$. A typical signed graph is the Seifert graph $G(D)$ associated to a knot diagram D: for each Seifert circle of D, we associate a vertex of $G(D)$ and two vertices of $G(D)$ are connected by an edge if there is a crossing of D whose adjacent two Seifert circles are corresponding to the two vertices. Each edge of $G(D)$ is labeled $+$ or $-$ depending on the sign of its associated crossing of D. For convenience, we say a $+$ or $-$ edge instead of an edge labeled $+$ or $-$.

A *block* of a (signed) graph is a maximal subgraph of the graph with no cut-vertices. A signed graph is *homogeneous* if each block has the same signs. A diagram D of a knot is *homogeneous* if $G(D)$ is homogeneous. Cromwell [4] showed that alternating diagrams and positive diagrams are homogeneous. There are many homogeneous diagrams which are non-alternating and non-positive.

Example 2.1. *Let D be the non-alternating and non-positive diagram as in Figure 1. Then $G(D)$ is homogeneous (see Figure 2). Therefore D is a homogeneous diagram which is non-alternating and non-positive. Note that D is not minimal crossing diagram (this is not used later).*

[3]In this paper, we use the notation "cycle" instead of "circuit".

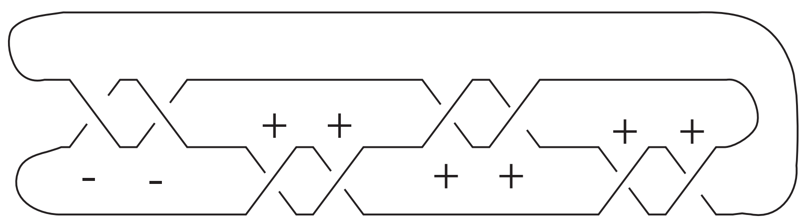

Figure 1: a non-alternating and non-positive diagram

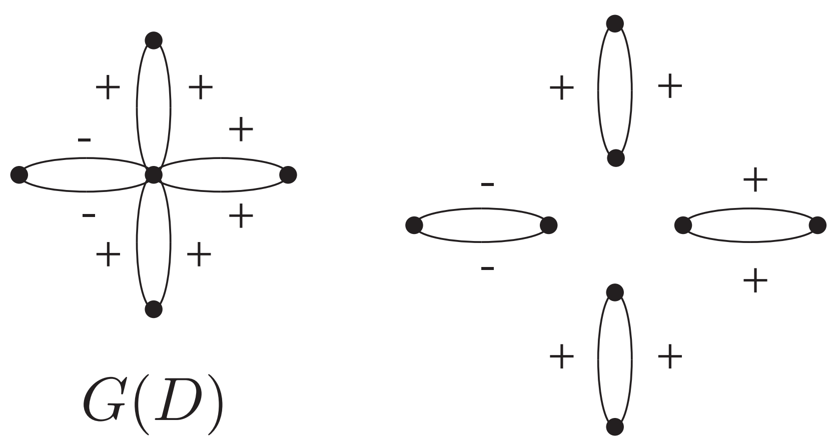

Figure 2: the graph $G(D)$ is homogeneous

Let B_n be the braid group on n strands with generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$. Stallings [31] introduced the notion of a homogeneous braid. A braid $\beta = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\cdots\sigma_{i_k}^{\epsilon_k}, \epsilon_j = \pm 1$ $(j = 1, \cdots, k)$ is *homogeneous* if

(1) every σ_j occurs at least once,

(2) for each j, the exponents of all occurrences of σ_j are the same.

For example, the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ is homogeneous, however, the braid $\sigma_1^2\sigma_2\sigma_1\sigma_2^{-1}$ is not homogeneous. Stallings [31] proved that the closure of a homogeneous braid is fibered. The following lemma is origin of the name "homogeneous".

Lemma 2.2 ([4])**.** *Let β be a braid whose closure is a knot. Then β is homogeneous if and only if the braid diagram of the closure of β is homogeneous.*

A knot K is *homogeneous* if K has a homogeneous diagram. The class of homogeneous knots includes alternating knots and positive knots. There are homogeneous knots which are non-alternating and non-positive and Cromwell [4] showed that the knot 9_{43} is the simplest one. One of the distinguished properties of a homogeneous diagram is the following.

Theorem 2.3 ([4])**.** *Let D be a homogeneous diagram of a knot K. Then the genus of K is realized by that of the Seifert surface obtained by applying Seifert's algorithm to D.*

Cromwell proved the above theorem algebraically. There is a geometric proof. Here we give an outline of the proof, which is suggested by M. Hirasawa.

The Seifert circles of a diagram is divided into two types: a Seifert circle is *of type 1* if it does not contain any other Seifert circles in $\mathbb{R}^2$, otherwise it is *of type 2*. Let $D \subset \mathbb{R}^2$ be a knot diagram and C a type 2 Seifert circle of D. Then C separates $\mathbb{R}^2$ into two components U and V such that $U \cup V = \mathbb{R}^2$ and $U \cap V = \partial U = \partial V = C$. Let D_1 and D_2 be the diagrams formed form $D \cap U$ and $D \cap V$ by adding suitable arcs from C respectively. If both $(U - C) \cap D \neq \emptyset$ and $(V - C) \cap D \neq \emptyset$, then C *decomposes* D into a $*$-product of D_1 and D_2, which is denoted by $D = D_1 * D_2$. Then the Seifert surface obtained by applying Seifert's algorithm to D is a Murasugi sum of Seifert surfaces obtained by applying Seifert's algorithm to D_1 and D_2 respectively (for the definition of a Murasugi sum, see [14] or [8]). A diagram is *special* if D has no decomposing Seifert circles of type 2. A special positive (or negative) diagram is alternating. Cromwell implicitly showed the following (see Theorem 1 in [4]).

Lemma 2.4 ([4])**.** *Let D be a homogeneous diagram of a knot K. Then*
(1) there are special diagrams $D_1, \cdots, D_n$ such that $D = D_1 * D_2 * \cdots * D_n$,
(2) each special diagram D_i $(i = 1, \cdots, n)$ is the connected sum of special alternating diagrams,
(3) each special alternating diagram corresponds to a block of $G(D)$.

Let D be homogeneous diagram of a knot K. Then, by Lemma 2.4, the Seifert surface S obtained by applying Seifert's algorithm to D is Murasugi sums of the Seifert surfaces obtained by applying Seifert's algorithm to the special alternating diagrams. The following lemma is classical results of Crowell and Murasugi.

Lemma 2.5. *Let D be a alternating diagram of a knot K. Then the genus of K is realized by that of the Seifert surface obtained by applying Seifert's algorithm to D.*

In [9], Gabai gave an elementary proof of Lemma 2.5 by using cut-and-past arguments. By Lemma 2.5, S is Murasugi sums of minimal Seifert surfaces. Let R_1 and R_2 be two minimal Seifert surfaces. Then a Murasugi sum of R_1 and R_2 is a minimal Seifert surface due to Gabai [8]. Therefore we obtain a geometric proof of Theorem 2.3.

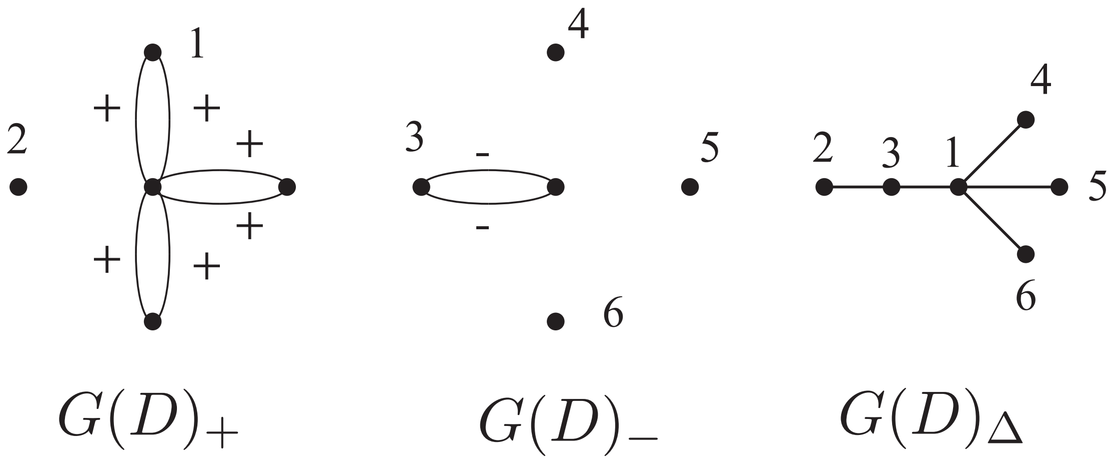

Figure 3: the graph G_Δ is tree

3 A characterization of a homogeneous diagram

In this section, we give a new characterization of a homogeneous diagram (Theorem 3.4). In particular, we show that if D is a homogeneous diagram of a knot, then $\Delta(D) = 0$. One can prove this by induction on the number of cut-vertices of $G(D)$, however, we prove this more graph-theoretically. For the Seifert Graph $G(D)$ associated to a knot diagram D, we construct a graph (which is denoted by $G(D)_\Delta$ later) such that the number of cycles of the graph is equal to $\Delta(D)$ and we prove if D is homogeneous, then the graph has no cycles. By using Theorems 1.1 and 3.4, we determine the Rasmussen invariant of a homogeneous knot (Theorem 1.2).

Let G_+ and G_- be the graphs which are obtained from a signed graph G by removing all $-$ and $+$ edges, respectively. Here we note that, by definition, each vertex of G belongs to exactly one connected component of G_+ and G_- respectively. Let G_Δ be the graph whose vertices are the connected components of G_+ and G_- and two vertices of G_Δ are connected by an edge if a vertex of G belong to the two connected components (which correspond to the two vertices). We give two examples, which provide us the idea of the proof of Lemma 3.3.

Example 3.1. *Let $G(D)$ be the signed graph as in Figure 2. We label 1 and 2 the connected components of G_+ and 3, 4, 5 and 6 the connected components of G_-. Then G_Δ is the graph as in Figure 3 and it is tree.*

Example 3.2. *Let G be the signed graph as in Figure 4. Then G has only one block and it is G itself. Since the block contains $+$ and $-$ edges, G is not homogeneous. Note that G has a cycle which contains $+$ and $-$ edges (in this case, the cycle is unique). We label 1 and 2 the connected components of G_+ and 3, 4, 5 and 6 the connected components of G_-. Then G_Δ is the graph as in Figure 4 and G_Δ has a cycle which is denoted by $(1,5)(5,2)(2,6)(6,1)$ (in this case, the cycle is also unique).*

Conversely, let $\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3$ and $\widetilde{e}_4$ be the edges of G_Δ and $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3, \widetilde{v}_4$and $\widetilde{v}_5$ the vertices of G_Δ as in Figure 5. Let v_i $(i = 1, \cdots, 4)$ be the vertex of G which corresponds to $\widetilde{e}_i$ and let $v_5 = v_1$. Then $\widetilde{v}_{i+1}$ as a connected component of G_+ or G_- contains v_i and v_{i+1} and there exists a simple path l_i in $\widetilde{v}_{i+1}$ from v_i to v_{i+1}. Therefore we obtain a cycle $l_1l_2l_3l_4$ from v_1 to $v_5(= v_1)$.

For a signed graph G, we denote by $\mathrm{sign}(e)$ the sign of an edge e of G. We show the following lemma to prove Theorem 3.4. To prove (2) $\Longleftarrow$ (3) is essential.

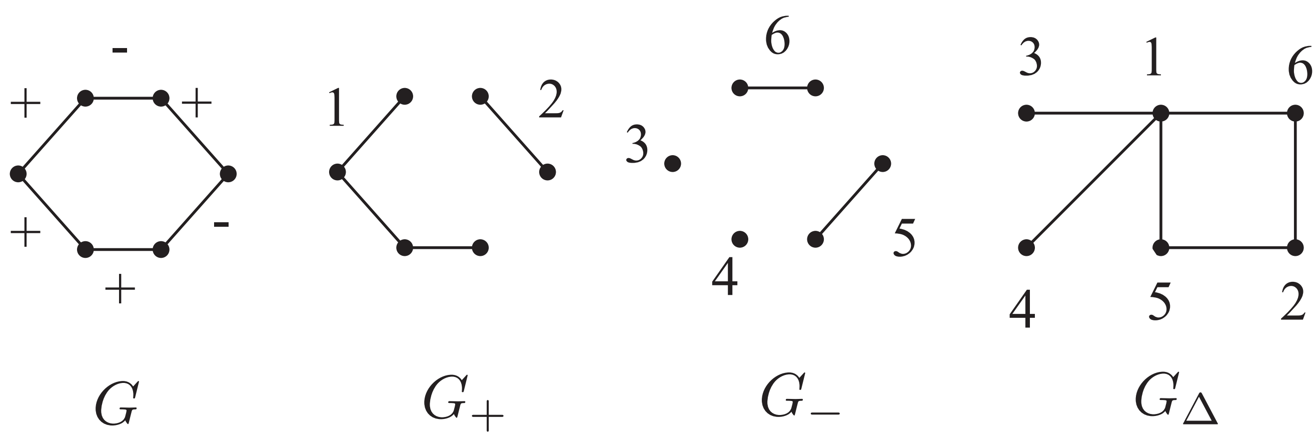

Figure 4: the graph G is not homogeneous

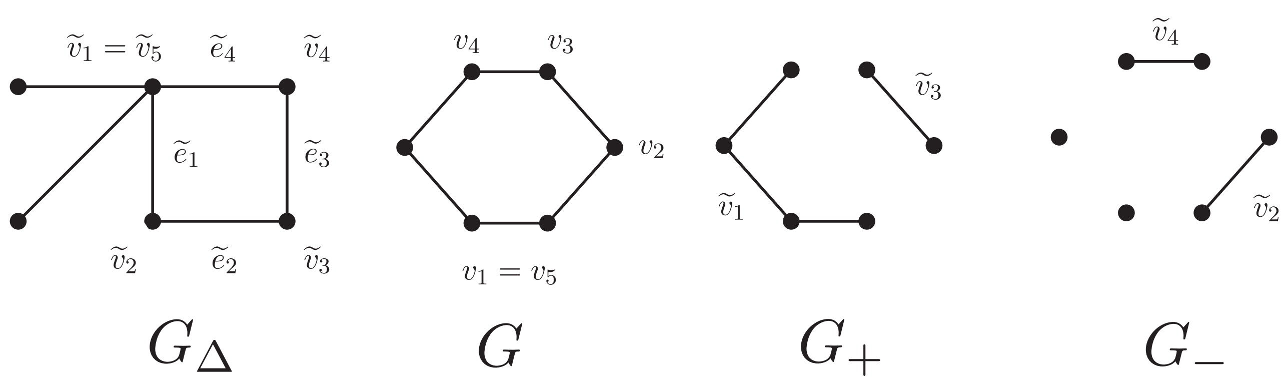

Figure 5: the graph G is not homogeneous

Lemma 3.3. *Let G be a signed graph. The following are equivalent.*
(1) G is not homogeneous.
(2) G has a cycle which contains both $+$ and $-$ edges.
(3) G_Δ has a cycle.

Proof. (1) $\Longrightarrow$ (2) Since G is not homogeneous, by definition, there exists a block which contains $+$ and $-$ edges. Then there exist a vertex v and edges e_1 and e_2 of the block such that one of the endpoints of e_1 and e_2 is v respectively and $\mathrm{sign}(e_1) \neq \mathrm{sign}(e_2)$. Now v is not a cut-vertex since v is a vertex of the block. There G has a cycle which contains both $+$ and $-$ edges.
(1) $\Longleftarrow$ (2) Let $e_1 \cdots e_n$ be the cycle of G which contains both $+$ and $-$ edges. Then there exists a natural number i such that $\mathrm{sign}(e_i) \neq \mathrm{sign}(e_{i+1})$. Let v be the vertex such that one of the endpoints of e_i and e_{i+1} is v respectively. Since v is not a cut-vertex, edges e_i and e_{i+1} belong to the same block. Therefore G is not homogeneous.
(2) $\Longrightarrow$ (3) Let $(e_1 \cdots e_{i_1}) \cdots (e_{i_{k-1}+1} \cdots e_{i_k}) \cdots (e_{i_{j-1}+1} \cdots e_{i_j})$ be the cycle which contains both $+$ and $-$ edges, where $\mathrm{sign}(e_{i_{k-1}+1}) = \cdots = \mathrm{sign}(e_{i_k})$ and $\mathrm{sign}(e_{i_k}) \neq \mathrm{sign}(e_{i_k+1})$. Then the path $e_{i_{k-1}+1} \cdots e_{i_k}$ is also one in $G_{\mathrm{sign}(e_{i_k})}$, which contracts to a vertex $\widetilde{v}_k$ of G_Δ $(k = 1, \cdots, j)$. Let $\widetilde{e}_k$ be the edge of G_Δ whose endpoints are $\widetilde{v}_k$ and $\widetilde{v}_{k+1}$ $(k = 1, \cdots, j-1)$, which is corresponding to the vertex v_k such that one of the endpoints of e_{i_k} and e_{i_k+1} is v_k respectively. Let $\widetilde{e}_j$ be the edge of G_Δ whose endpoints are $\widetilde{v}_j$ and $\widetilde{v}_1$, which is corresponding to the vertex v_j such that one of the endpoints of e_{i_j} and e_1 is v_j respectively.

Therefore $\widetilde{e}_1 \cdots \widetilde{e}_j$ is a path from $\widetilde{v}_1$ to $\widetilde{v}_1$, possibly not a cycle. If the path is not a cycle, we choose a cycle (as a subsequence of edges of the path). Therefore G_Δ has a cycle.
(2) $\Longleftarrow$ (3) Let $\widetilde{e}_1 \cdots \widetilde{e}_n$ be the cycle G_Δ $(i = 1, \cdots, n)$ and denote $\widetilde{e}_i = (\widetilde{v}_i, \widetilde{v}_{i+1})$. Then $\widetilde{v}_{n+1} = \widetilde{v}_1$. Let v_i be the vertex of G which corresponds to $\widetilde{e}_i$ $(i = 1, \cdots, n)$ and $v_{n+1} = v_1$. Recall that a vertex of G_Δ corresponds to a connected component of G_+ or G_-. Then $\widetilde{v}_{i+1}$ (as a connected component of G_+ or G_-) contain v_i and v_{i+1} $(i = 1, \cdots, n)$. There exists a simple path l_i from v_i to v_{i+1}. There we obtain a path $l_1 l_2 \cdots l_n$ from v_1 to $v_{n+1} (= v_1)$, possibly not a cycle. If the path is not a cycle, we choose a cycle (as a subsequence of edges of the path). By the construction, the cycle always contains both $+$ and $-$ edges. $\square$

Note that $O_+(D)$ and $O_-(D)$ are equal to the numbers of connected components of $G(D)_+$ and $G(D)_-$, respectively. Therefore the number of vertices of $G(D)_\Delta$ is equal to $O_+(D) + O_-(D)$ and, by definition, the number of edges of $G(D)_\Delta$ is equal to $O(D)$. Lobb [20] showed that $\Delta(D) = b_1(G(D)_\Delta)$ for any diagram D. For the completeness, we recall the proof here.

$$\begin{aligned} b_1(G(D)_\Delta) &= b_0(G(D)_\Delta) - \chi(G(D)_\Delta) \\ &= 1 - (O_+(D) + O_-(D) - O(D)) \\ &= \Delta(D), \end{aligned}$$

where b_i denotes the i-th Betti number $(i = 0, 1)$ and χ denotes the Euler characteristic. Then we obtain the following.

Theorem 3.4. *A diagram D of a knot is homogeneous if and only if $\Delta(D) = 0$.*

Proof. By the above argument, $\Delta(D) = 0$ if and only if G_Δ is tree. Therefore the proof immediately follows from Lemma 3.3. $\square$

Now we prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 3.4, we obtain $\Delta(D) = 0$. As mentioned before, Lobb [20] showed that if $\Delta(D) = 0$, then $s(K) = w(D) - O(D) + 2O_+(D) - 1$. This completes the proof. $\square$

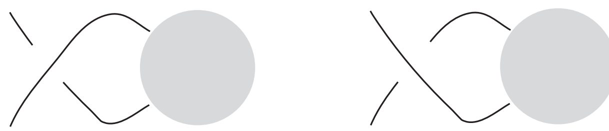

Figure 6: nugatory crossings

4 Positivity of knots

In this section, we recall some notions of positivity and give some characterizations of a positive knot (Theorem 1.3). In particular, we recover Baader's theorem which states that a knot is positive if and only if it is homogeneous and strongly quasipositive.

Let D be a diagram of a knot. We denote by D_p the diagram which is obtained from D by smoothing (along the orientation of D) at a crossing p. A crossing of D is *nugatory* if there exists a curve l such that the intersection of D and l is only a crossing of D (see also Figure 6). Then it is easy to see that the following lemma holds.

Lemma 4.1. *Let p be a crossing of D. Then p is nugatory if and only if the number of the connected components of D_p is two.*

Here we recall some notions of positivity for knots. A knot is *braid positive* if it is the closure of a braid of the form $\beta = \prod_{k=1}^{m} \sigma_{i_k}$. A knot is *positive* if it has a diagram without negative crossings. L. Rudolph introduced the concept of a (strongly) quasipositive knot (see [27]). Let

$$\sigma_{i,j} = (\sigma_i, \cdots, \sigma_{j-2})(\sigma_{j-1})(\sigma_i, \cdots, \sigma_{j-2})^{-1}.$$

A knot is *strongly quasipositive* if it is the closure of a braid of the form

$$\beta = \prod_{k=1}^{m} \sigma_{i_k, j_k}.$$

A knot is *quasipositive* if it is the closure of a braid of the form

$$\beta = \prod_{k=1}^{m} \omega_k \sigma_{i_k} \omega_k^{-1},$$

where ω_k is a word in B_n. The following are known.

(1) Let K be a torus knot.[4] Then $\tau(K) = s(K)/2 = g_*(K) = g(K)$, where $g(K)$ denotes the (Seifert) genus of K. This is due to Rasmussen for s [25] and Ozsváth and Szabó for τ [22]. These equalities provide a proof of the Milnor conjecture.

(2) Let K be a strongly quasipositive knot. Then $\tau(K) = s(K)/2 = g_*(K) = g(K)$. This is due to Livingston [17].

(3) Let K be a quasipositive knot. Then $\tau(K) = s(K)/2 = g_*(K)$. This is due to Plamenevskaya [23] and Hedden (with a detailed and constructive proof) [10] for τ, and Plamenevskaya [24] and Shumakovitch [30] for s.

[4] A torus knot is strongly quasipositive.

By using Lemma 4.1, we prove Theorem 1.3.

Proof of Theorem 1.3. (1) $\Longrightarrow$ (2) A positive knot is strongly quasipositive ([21] and [28]).
(2) $\Longrightarrow$ (3) A strongly quasipositive knot K is a quasipositive knot with $g_*(K) = g(K)$ [27].
(3) $\Longrightarrow$ (4) Since K is a quasipositive knot, $\tau(K) = s(K)/2 = g_*(K)$. By the assumption, $g_*(K) = g(K)$. Therefore $\tau(K) = s(K)/2 = g_*(K) = g(K)$.
(4) $\Longrightarrow$ (1) Let D be a homogeneous diagram of K. Then the genus of K is realized by that of the surface constructed by applying Seifert's algorithm to D (Theorem 2.3). Therefore $2g(K) = 1 + c(D) - O(D)$, where $c(D)$ denotes the number of crossings of D. By Theorem 1.2, we have $s(K) = w(D) - O(D) + 2O_+(D) - 1$. By assumption, $s(K) = 2g(K)$. This implies that $O_+(D) - 1 = n_-(D)$, where $n_-(D)$ denotes the number of negative crossings of D.

If there exists a non-nugatory negative crossing p of D, then D_p is connected by Lemma 4.1. Therefore $O_+(D) - 1 < n_-(D)$ (since, in general, the difference of the numbers of the connected components of two link diagrams D_1 and D_2 such that D_2 is obtained from D_1 by smoothing at a crossing of D_1 is 0 or 1). This contradicts the fact that $O_+(D) - 1 = n_-(D)$. Therefore all negative crossings of D are nugatory and D represents a positive knot. $\square$

Corollary 1.4 immediately follows from Theorem 1.3.

5 A new approach to estimate the Rasmussen invariant of a knot

Let D be a diagram of a knot with $\Delta(D) \neq 0$. Then Kawamura-Lobb's inequality may not be sharp as follow.

Example 5.1. *Let K be the pretzel knot of type $(3, -5, -7)$ and D the standard pretzel diagram of K. Then $\omega(D) = 9, O(D) = 14, O_+(D) = 3$ and $O_-(D) = 11$. Therefore $\Delta(D) = 1$ and $w(D) - O(D) + 2O_+(D) - 1 = 0$. On the other hand, since K is strongly quasipositive [29],[5] we obtain $s(K) = 2g_*(K) = 2g(K) = 2$. Remark that K is topologically slice but not smoothly slice.*

We need a more shaper estimation to describe the Rasmussen invariant of the pretzel knot of type $(3, -5, -7)$ in terms of its standard pretzel diagram. Roughly speaking, there are two approaches to estimate or determine the Rasmussen invariant. One of them is to compute the Khovanov homology by using a computer and to use the spectral sequence which converges to Lee's homology. The other is to use some formal properties of the Rasmussen invariant (and the tau invariant). We propose a new and direct approach to estimate or determine the Rasmussen invariant. We briefly recall the definition of the Rasmussen invariant to explain this. For a full explanation, see [25].

Let D be a diagram of a knot K and $C^*_{Lee}(D)$ Lee's complex (see [25] for the definition). Then Lee [16] proved that the homology group of $C^*_{Lee}(D)$ is independent of the choice of diagrams of K. Lee's homology of K, denoted by $H^*_{Lee}(K)$, is defined to be the homology group of $C^*_{Lee}(D)$. In addition, for a diagram D of a knot K, Lee [16] associated two (co)cycles of $C^*_{Lee}(D)$, denoted by f_o and $f_{\bar{o}}$,[6] and proved that $[f_o]$ and $[f_{\bar{o}}]$ are a basis of $H^*_{Lee}(K)$, in particular, that the dimension of $H^*_{Lee}(K)$ is equal to two, where $[\cdot]$ denotes its homology class. This basis is called canonical since the basis is determined up to multiple of 2^c for K [25], where c is an integer.

Rasmussen [25] defined a filtration grading q on a non-zero element of $C^*_{Lee}(D)$ (which induces a filtration on $C^*_{Lee}(D)$). Then a filtration grading s on a non-zero element $[x]$ of $H^*_{Lee}(K)$ (which

[5] In [29], K is denoted by $P(-3, 5, 7)$. Our notation is the same as that in [5] and [14].
[6] In [25], these are denoted by s_o and $s_{\bar{o}}$ respectively

also induces a filtration on $H^*_{Lee}(K)$) is defined as follows.

$$s([x]) = \max\{q(y) | [x] = [y]\}.$$

Then the Rasmussen invariant of K, denoted by $s(K)$, is defined to be $s([f_o]) + 1 (= s([f_{\bar{o}}]) + 1)$.

Since $s([f_o]) \geq q(f_o)$ and $q(f_o) = \omega(D) - O(D)$ (by the definition of q), we obtain $s(K) \geq \omega(D) - O(D) + 1$. This is the slice-Bennequin inequality for the Rasmussen invariant for K (see [24] and [30]). Theorem 1.1 implies that there exists a cycle f such that $[f_o] = [f]$ and $q(f) = \omega(D) - O(D) + 2O_+(D) - 2$, however, yet no one has succeeded to describe f explicitly. In [1], as a first step toward this, we describe a cycle f_1 with $[f_o] = [f_1]$ which gives the so-called shaper slice-Bennequin inequality for the Rasmussen invariant of a knot [12] (which is stronger than the slice-Bennequin inequality and weaker than the inequality of Kawamura and Lobb, see [13]). In the future work, the graph $G(D)_\Delta$ is expected to play an important role (see also [6]). We conclude this paper by giving the following problem.

Problem. Let D be the standard diagram of $P(3, -5, -7)$. Find a cycle f of $C^*_{Lee}(D)$ such that $[f_o] = [f]$ and $q(f) = \omega(D) - O(D) + 2O_+(D) = 1$.

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